Exercise 61

(a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + gfh'$.

(b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $y = e^{3x}$.

Solution

Use the product rule twice to find the derivative of fgh.

$$\begin{aligned}(fgh)' &= \frac{d}{dx}\bigg[f(x)g(x)h(x)\bigg] \\ &= \frac{d}{dx}\left\{\bigg[f(x)g(x)\bigg]h(x)\right\} \\ &= \frac{d}{dx}\bigg[f(x)g(x)\bigg]h(x) + \bigg[f(x)g(x)\bigg]h'(x) \\ &= \bigg[f'(x)g(x) + f(x)g'(x)\bigg]h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)\end{aligned}$$

Set $f = g = h$ in this result.

$$\begin{aligned}(fff)' = \frac{d}{dx}[f(x)]^3 &= f'(x)f(x)f(x) + f(x)f'(x)f(x) + f(x)f(x)f'(x) \\ &= 3[f(x)]^2 f'(x)\end{aligned}$$

This can be applied to differentiate $y = e^{3x}$.

$$\begin{aligned}y' &= \frac{d}{dx}(e^{3x}) \\ &= \frac{d}{dx}(e^x)^3 \\ &= 3(e^x)^2\frac{d}{dx}(e^x) \\ &= 3e^{2x}e^x \\ &= 3e^{2x+x} \\ &= 3e^{3x}\end{aligned}$$